

Faddeev-Jackiw Quantization of Non-Autonomous Singular Systems

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We extend the quantization à la Faddeev-Jackiw for non-autonomous singular systems. This leads to a generalization of the Schrödinger equation for those systems. The method is exemplified by the quantization of the damped harmonic oscillator and the relativistic particle in an external electromagnetic field.

PACS numbers:

The quantization of constrained systems is almost as old as the beginning of quantum mechanics. It was Dirac [1] who elaborated a Hamiltonian approach with a categorization of constraints and the introduction of the so-called Dirac brackets. Later, Faddeev and Jackiw [2] suggested an alternative and generally simpler method based on a symplectic structure. Recently, we have proposed a third approach for classically soluble constrained systems where the brackets between the constants of integration are computed. This method does neither require Dirac formalism nor the symplectic method of Faddeev-Jackiw [3]. All three approaches were developed for autonomous constrained systems only. The quantization of non-autonomous singular systems has turned to be non trivial [4]. Gitman and Tyutin [5], via notably the introduction of a conjugate momentum of time, could extend Dirac approach and brackets for those systems. In the present work, our aim is to generalize the Faddeev-Jackiw symplectic approach to non-autonomous constrained systems. This leads to a generalization of the Schrödinger equation which encompasses non-autonomous singular systems. The quantization of a relativistic particle in an electromagnetic field is solved by this method, constituting an original derivation of the Dirac equation.

Consider a non autonomous Lagrangian one-form as in [2]

$$Ldt = a_j(\xi, t)d\xi^j - H(\xi, t)dt, \quad (1)$$

where ξ^j are the N -component phase-space coordinates with $j = 1, \dots, N$, where a_j and H are time-dependent. The Euler-Lagrange equations lead to

$$\dot{\xi}^i = f_{ij}^{-1} \left(\frac{\partial H}{\partial \xi^j} + \frac{\partial a_j}{\partial t} \right) \quad (2)$$

$$f_{ij} = \frac{\partial a_j}{\partial \xi^i} - \frac{\partial a_i}{\partial \xi^j} \quad (3)$$

for an invertible antisymmetric matrix f_{ij} . The dot denotes differentiation with respect to t . Eq. (2) can not be derived from Hamilton equations through brackets when $\partial a_j / \partial t \neq 0$ (see Eq. (17)). Canonical quantization seems compromised in this case. To solve this problem via the Faddeev-Jackiw approach we first introduce a time parameter τ such that time is promoted to a dynamically

variable $t = t(\tau)$ and $\xi^i = \xi^i(\tau)$. This leads to a Lagrangian L_τ , given by $L_\tau d\tau = Ldt$, so that the action remains the same. L_τ has a gauge invariance due to the arbitrariness of the parameter τ . We thus define a new Lagrangian $\tilde{L}_\tau = L_\tau + \omega(t' - 1)$ that implements the gauge constraint $t' = 1$ (prime denotes differentiation with respect to τ) via $\omega(\tau)$ a Lagrange multiplier seen as a new variable. Dropping a total time derivative term, one ends up with the following Lagrangian one-form

$$\tilde{L}d\tau = a_j(\xi, t)d\xi^j - H(\xi, t)dt - t d\omega - \tilde{H}d\tau, \quad (4)$$

where $\tilde{H}(\xi, t, \omega) = \omega$ defines a new Hamiltonian. The Euler-Lagrange equations with \tilde{L} lead to Eq (2) and the equations $\omega' = 0$ and $t' = 1$. Unlike the initial Lagrangian (1), \tilde{L} is autonomous with respect to τ and Eq. (4) is precisely of the form studied by Faddeev-Jackiw [2]

$$\tilde{L}d\tau = A_i(\zeta)d\zeta^i - \tilde{H}(\zeta)d\tau, \quad (5)$$

with now ζ^i the $N+2$ -component phase space coordinates defined by $\zeta^i = \xi^i$ for $i = 1 \dots N$; $\zeta^{N+1} = t$ and $\zeta^{N+2} = \omega$. Here $A_i = a_i$ for $i = 1 \dots N$ and $A_{N+1} = -H$, $A_{N+2} = -t$.

Following [2] we first consider the Euler-Lagrange equations

$$F_{ij}\zeta^{j'} = \frac{\partial \tilde{H}}{\partial \zeta^i} \quad (6)$$

with

$$F_{ij} = \frac{\partial A_j}{\partial \zeta^i} - \frac{\partial A_i}{\partial \zeta^j}. \quad (7)$$

If the antisymmetric matrix F_{ij} is regular then Eq. (6) becomes

$$\zeta^{i'} = F_{ij}^{-1} \frac{\partial \tilde{H}}{\partial \zeta^j}. \quad (8)$$

Writing the Hamilton equations

$$\zeta^{i'} = \left\{ \zeta^i, \tilde{H} \right\} = \left\{ \zeta^i, \zeta^j \right\} \frac{\partial \tilde{H}}{\partial \zeta^j} \quad (9)$$

the generalized brackets for non-autonomous systems are readily obtained as

$$\{\zeta^i, \zeta^j\} = F_{ij}^{-1} \quad i, j = 1 \dots N+2. \quad (10)$$

Now from the matrix F_{ij}^{-1} we obtain the following fundamental brackets

$$\{\xi^i, \xi^j\} = f_{ij}^{-1} \quad i, j = 1 \dots N \quad (11)$$

$$\{\xi^i, t\} = 0, \quad (12)$$

$$\{t, \omega\} = 1, \quad (13)$$

$$\{\xi^i, \omega\} = f_{ij}^{-1} \left(\frac{\partial a_j}{\partial t} + \frac{\partial H}{\partial \xi^j} \right). \quad (14)$$

The brackets $\{\xi^i, \xi^j\} = f_{ij}^{-1}$ which are unchanged with respect to the autonomous case are sufficient to deduce all brackets of the theory. We see that ω plays the role of a conserved Hamiltonian for the time evolution of ξ^i . Indeed, let us now use the fact that $\tilde{H}(\xi, t, \omega) = \omega$. Eq.(9) then reduces to the "Hamilton" equations of motion

$$\frac{d\xi^i}{d\tau} = \{\xi^i, \omega\} \quad i = 1 \dots N+2. \quad (15)$$

For $i = N+2$ and $N+1$, Eq. (15) gives respectively $d\omega/d\tau = 0$ so ω is a constant and $dt/d\tau = 1$, so $t(\tau) = \tau$ without loss of generality. The dynamics of the phase-space coordinates $\xi^i(t)$ with respect to the physical time t is therefore

$$\dot{\xi}^i = \{\xi^i, \omega\} = f_{ij}^{-1} \frac{\partial H}{\partial \xi^j} + f_{ij}^{-1} \frac{\partial a_j}{\partial t}, \quad (16)$$

for $i = 1 \dots N$. These equations are the same as the Euler-Lagrange equations Eq. (2). Eq. (16) is the generalization of the Hamilton equation for a non-autonomous system. Indeed the equations of motion can not be obtained by usual Hamilton equations which would give instead

$$\dot{\xi}^i = \{\xi^i, H\} = f_{ij}^{-1} \frac{\partial H}{\partial \xi^j} \quad (17)$$

correct for the case $\partial a_j / \partial t = 0$ only.

The time derivative of any function $O(\xi^i, t) = \xi^j \partial O / \partial \xi^j + \partial O / \partial t$ is also obtained as

$$\dot{O}(\xi^i, t) = \{O, \omega\}. \quad (18)$$

This is a new general relation that embraces all systems, regular and autonomous as well. It does not contain the term $\partial O / \partial t$. The Hamiltonian H is generally not conserved for a non-autonomous system contrary to ω and $\dot{H} = \{H, \omega\} = -\xi^j \partial a_j / \partial t + \partial H / \partial t (= -\partial L / \partial t)$.

Before considering the quantization, we note from Eq. (6) that the condition of regularity of the matrix F_{ij} is actually reduced to the regularity of f_{ij} . Now, for a singular matrix f_{ij} we recall briefly the procedure which

consists first in determining the zero-modes $v^{(\alpha)}$, which are solutions of $v_i^{(\alpha)} f_{ij} = 0$. Multiplying Eq. (16) by $v_i^{(\alpha)}$ we obtain the equation $v_i^{(\alpha)} (\partial H / \partial \xi_i + \partial a_i / \partial t) = 0$ where we used the equations $\partial \tilde{H} / \partial \omega = 1$ and $\partial \tilde{H} / \partial \xi_i = \partial \tilde{H} / \partial t = 0$. These relations between the variables ξ_i and t constitute a set of constraints $\phi_\alpha(\xi, t) = 0$ that must be conserved in time $d\phi_\alpha / d\tau = 0$. We therefore add to the Lagrangian \tilde{L} the term $\lambda'_\alpha \phi_\alpha$ to obtain the new autonomous Lagrangian $\tilde{L} + \lambda'_\alpha \phi_\alpha$, and redo the Faddeev-Jackiw procedure considering λ_α as new independent variables. Now, if the new matrix f_{ij} is invertible, all brackets are accessible. If not we calculate the zero-modes that will give in principle new constraints that we should add to the Lagrangian and so on. At the end, either we obtain an invertible matrix and we can determine the brackets, or the matrix is always singular with no new constraints. In this case, our initial Lagrangian has gauge symmetry that should be fixed by using additional conditions to obtain a non singular matrix f_{ij} .

Quantization. We first remark that from Eqs. (16) and (17) ω can be decomposed as $\omega = \varepsilon + H$ with the brackets

$$\{\xi^i, H\} = f_{ij}^{-1} \frac{\partial H}{\partial \xi^j} \quad \text{and} \quad \{\xi^i, \varepsilon\} = f_{ij}^{-1} \frac{\partial a_j}{\partial t}. \quad (19)$$

Eqs. (12) and (13) show that $\{t, \varepsilon\} = 1$ so ε is a variable conjugate to time. It was introduced in [5] for singular systems and in [6] for regular ones. Although it is not suitable to talk about momentum conjugate in the Faddeev-Jackiw approach it is nevertheless interesting to see that ε is actually equal to the conjugate momentum of time defined as $p_t(t) = (\partial \tilde{L}_\tau / \partial t')_{\tau=t}$. Thus $\omega = p_t + H(p, q, t)$ can be interpreted as an extended Hamiltonian in an extended phase space [7].

For the quantization in the Schrödinger picture we define the operators $\hat{\zeta}^i$ associated to their classical counterparts. Any operator \hat{O} associated to a classical function $O(\zeta)$ is defined by the rule $\hat{O} = O(\hat{\zeta})$ and commutators are defined as $[\cdot, \cdot] = i\hbar \{\cdot, \cdot\}_{\zeta=\hat{\zeta}}$ (we disregard problems with operator ordering). Therefore from Eq. (10) we have

$$[\hat{\zeta}^i, \hat{\zeta}^j] = i\hbar \{\zeta^i, \zeta^j\}_{\zeta=\hat{\zeta}} = i\hbar F_{ij}^{-1}(\hat{\zeta}), \quad (20)$$

thus in general for singular systems operators do not satisfy the canonical commutation relations. An operator \hat{O} associated to $O(\xi, t)$ will be given by the quantum version of Eq. (18)

$$\hat{O} = \frac{1}{i\hbar} [\hat{O}, \hat{\omega}] = \{O, \omega\}_{\zeta=\hat{\zeta}}, \quad (21)$$

where $\hat{\omega} = \hat{\varepsilon} + \hat{H}$. As $[\hat{t}, \hat{\varepsilon}] = i\hbar$, we define naturally $\hat{t} = t$ and thus $\hat{\varepsilon} = -i\hbar d/dt$ is the time translation operator [12]. Therefore $d\hat{O}/dt = \frac{1}{i\hbar} [\hat{O}, \hat{\varepsilon}]$ and \hat{O} can also be

written

$$\hat{O} = \frac{1}{i\hbar}[\hat{O}, \hat{H}] + \frac{d\hat{O}}{dt}, \quad (22)$$

which is different from the usual expression $\hat{O} = \frac{1}{i\hbar}[\hat{O}, \hat{H}] + \partial\hat{O}/\partial t$. The reason is that $\hat{\xi}^i$ is explicitly time dependent and its evolution is given by

$$\frac{d}{dt}\hat{\xi}^i = \{\xi^i, \varepsilon\}_{\xi=\hat{\xi}} = \left(f_{ij}^{-1} \frac{\partial a_j}{\partial t}\right)_{\xi=\hat{\xi}}. \quad (23)$$

The solution of this differential equation gives us $\hat{\xi}^i(t)$ with $\hat{\xi}^i(0) = \hat{\xi}_s^i$ the usual time independent Schrödinger operator satisfying $[\hat{\xi}_s^i, \hat{\xi}_s^j] = i\hbar f_{ij}^{-1}(\hat{\xi}_s)$ (note that this quantization at $t = 0$ could be performed at any arbitrary time [5]). The operator $\hat{\xi}^i$ associated to ξ^i is therefore

$$\hat{\xi}^i = \{\xi^i, H + \varepsilon\}_{\xi=\hat{\xi}} = \left(f_{ij}^{-1} \frac{\partial H}{\partial \xi^j} + f_{ij}^{-1} \frac{\partial a_j}{\partial t}\right)_{\xi=\hat{\xi}}. \quad (24)$$

Introducing a quantum state $|\psi(t)\rangle$ we see that $\langle\psi|\hat{O}|\psi\rangle = \frac{d}{dt}\langle\psi|\hat{O}|\psi\rangle$ only if $|\psi(t)\rangle$ satisfies the equation

$$\hat{\omega}|\psi(t)\rangle = 0 \quad (25)$$

with $\hat{\omega} = -i\hbar d/dt + \hat{H}$. Eq. (25) is a generalization of the Schrödinger equation and is the quantum evolution for all quantum systems including singular non-autonomous ones. The decomposition $\hat{\omega} = -i\hbar d/dt + H$ is not always valid, as for instance, for a relativistic Lagrangian $H = 0$ (see later on), but in general we can write $\hat{\omega} = -i\hbar d/dt + \hat{H}_{eff}(t)$. As a first check, consider the regular Lagrangian $L = \frac{m}{2}\dot{\mathbf{r}}^2 - U(\mathbf{r}, t)$ that we write $\tilde{L}d\tau = \mathbf{p}d\mathbf{r} - Hdt - td\omega - \omega d\tau$ with $H = \mathbf{p}^2/2m + U(\mathbf{r}, t)$. From Eq. (10) we get the canonical relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, and the commutators $[\hat{\mathbf{r}}, \hat{\omega}] = i\hbar\hat{\mathbf{p}}/m = i\hbar\hat{\mathbf{r}}$ and $[\hat{\mathbf{p}}, \hat{\omega}] = -i\hbar\nabla U = i\hbar\hat{\mathbf{p}}$. These commutators are satisfied for $\hat{\omega} = -i\hbar d/dt + \hat{H}$ as expected.

Note that here the time operator does commute with the Hamiltonian $[\hat{t}, \hat{H}] = 0$ but instead $[\hat{t}, \hat{\varepsilon}] = i\hbar$. This is physically correct as in quantum physics the energy can be measured with arbitrary precision at any time. The common idea that a time operator has to satisfy $[\hat{t}, \hat{H}] = i\hbar$, comes from the usually assumed relation $d\hat{t}/d\tau = \frac{1}{i\hbar}[\hat{t}, \hat{H}] = 1$ which is wrong (even for regular systems), instead $d\hat{t}/d\tau = \frac{1}{i\hbar}[\hat{t}, \hat{\omega}] = 1$. Therefore the problem of unboundedness of the energy spectrum [8] does not exist here.

In the Heisenberg picture $\hat{\xi}_H^i = U^{-1}\hat{\xi}^i U$ with $U(t)$ the time evolution operator $|\psi(t)\rangle = U|\psi\rangle_H$. Using Eqs. (23) and (25) we find

$$\frac{d\hat{\xi}_H^i}{dt} = \frac{1}{i\hbar}[\hat{\xi}_H^i, \hat{\omega}_H] = \{\xi^i, \omega\}_{\xi=\hat{\xi}_H, \omega=\hat{\omega}_H} \quad (26)$$

which is exactly given by Eq. (16) with ξ^i replaced by $\hat{\xi}_H^i$.

We will now apply the Faddeev-Jackiw quantization approach to two examples of non-autonomous systems.

Damped harmonic oscillator. Consider the following singular non autonomous Lagrangian introduced by [9]

$$L = \frac{1}{2}e^{2\alpha t}(y\dot{x} - x\dot{y} - y^2 - 2\alpha xy - \Omega^2 x^2). \quad (27)$$

In Dirac formulation this Lagrangian describes a singular system with time-dependent second-class constraints. The Euler-Lagrange equation corresponds to the damped harmonic oscillator whose quantization has been treated in [9] via the extended Dirac formalism for singular non autonomous system. With the Faddeev-Jackiw approach we start from the transformed Lagrangian one form

$$\tilde{L}d\tau = \frac{1}{2}e^{2\alpha t}(ydx - xdy) - Hdt - td\omega - \omega d\tau, \quad (28)$$

where $H = \frac{1}{2}e^{2\alpha t}(y^2 + 2\alpha xy + \Omega^2 x^2)$. The element of the 2×2 antisymmetric matrix $f_{x,y}^{-1}$ is easily computed and leads to

$$[\hat{x}, \hat{y}] = i\hbar e^{-2\alpha t}. \quad (29)$$

From Eq. (14) as usual $[\hat{t}, \hat{\omega}] = i\hbar$ and

$$[\hat{x}, \hat{\omega}] = i\hbar\hat{y} \quad (30)$$

$$[\hat{y}, \hat{\omega}] = -i\hbar(\Omega^2\hat{x} + 2\alpha\hat{y}). \quad (31)$$

Note that the whole set of brackets could be obtained at once by computing the full 4×4 matrix F_{ij}^{-1} directly. The commutator Eq. (29) is the same as in [9]. In the Schrödinger picture $\hat{t} = t$, $\hat{\varepsilon} = -i\hbar d/dt$ and from Eq. (23) we deduce $\hat{x}(t) = \hat{x}(0)e^{-\alpha t}$ and $\hat{y}(t) = \hat{y}(0)e^{-\alpha t}$. From Eq. (29) we see that $\hat{x}(0)$ and $\hat{y}(0)$ are conjugate to each other. Therefore, if we consider for instance the position representation we have $\hat{x}(t) = xe^{-\alpha t}$ and $\hat{y}(t) = -i\hbar e^{-\alpha t}\partial_x$ acting on the wave function $\psi(x, t)$. The commutators involving $\hat{\omega}$ leads to the natural choice

$$\hat{\omega} = -i\hbar\partial_t - \frac{1}{2}\hbar^2\partial_x^2 - i\alpha\hbar x\partial_x - \frac{i\alpha\hbar}{2} + \frac{\Omega^2}{2}x^2 \quad (32)$$

and the Schrödinger equation of the damped harmonic oscillator is $\hat{\omega}\psi(x, t) = 0$ which is the same equation than in [9]. This shows the equivalence of the two methods for the quantum damped harmonic oscillator.

Relativistic Lagrangian. We consider now the quantization of a relativistic point like particle in an external electromagnetic field. This system meets specific difficulties such as a null Hamiltonian and the presence of a gauge symmetry due to the arbitrary choice of the time parametrization [5, 10, 11]. We will see that the

Faddeev-Jackiw approach for non-autonomous systems can be straightforwardly applied and will lead to the Dirac equation. Consider a relativistic particle interacting with an electromagnetic potential $A^\mu = (\phi, \mathbf{A})$ whose quadri-position $x^\mu = x^\mu(\tau)$ with $\mu = 0..3$ depends on a parameter τ . The metric is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The reparametrization invariant action is $S = \int L_\tau d\tau$ with a Lagrangian one-form

$$L_\tau d\tau = -m ds - e A^\mu dx_\mu \quad (33)$$

in unit $c = 1$, with $ds = \sqrt{dx^\mu dx_\mu}$. The quadri-momenta $p_\mu = -m x'_\mu / s' - e A_\mu$ lead to a null Hamiltonian $H_\tau = p_\mu x'^\mu - L = 0$ as L is homogeneous of degree one in the velocities. Still we have the constraint

$$(p_0 + e\phi)^2 - (\mathbf{p} - e\mathbf{A})^2 = m^2. \quad (34)$$

By analogy to the historical calculus of Dirac, it can be linearized by introducing the usual Clifford algebra to get the equivalent constraint

$$\Lambda = p_0 + e\phi + \boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + \beta m = 0, \quad (35)$$

where the Clifford generators α_i ($i = 1, 2, 3$) and β satisfy the relations $\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}$, $\beta^2 = 1$ and $\alpha_i \beta + \beta \alpha_i = 0$. Dirac matrices are a four-dimensional representation of this Clifford Algebra. The constraint Eq. (35) is implemented by a Lagrange multiplier term $\lambda' \Lambda$ added to L_τ . The gauge invariance due to the arbitrary choice of τ of Eq. (33) is again fixed by adding $\omega(t' - 1)d\tau$ to L_τ (a condition equivalent to $t = \tau$) and we get the one-form Lagrangian

$$\tilde{L} d\tau = \mathbf{p} d\mathbf{x} + p_0 dt + \Lambda d\lambda - t d\omega - \omega d\tau. \quad (36)$$

It has the same form as Eq. (4) if we introduce the 10-component phase-space coordinates $\zeta^i = (\mathbf{x}, t, \mathbf{p}, p_0, \lambda, \omega)$. The matrix F_{ij}^{-1} can be easily computed and written formally as

$$F_{ij}^{-1} = \begin{pmatrix} 0 & 0 & 1 & -\boldsymbol{\alpha} & 0 & \boldsymbol{\alpha} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \mathbf{D} & 0 & -\mathbf{D} \\ \boldsymbol{\alpha} & 0 & -\mathbf{D} & 0 & -1 & -D_0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -\boldsymbol{\alpha} & -1 & \mathbf{D} & D_0 & 1 & 0 \end{pmatrix},$$

where $\mathbf{D} = e\nabla(\phi - \boldsymbol{\alpha} \cdot \mathbf{A})$ and $D_0 = e\partial_t(\phi - \boldsymbol{\alpha} \cdot \mathbf{A})$. From F_{ij}^{-1} we read off the non-vanishing brackets:

$$\{\mathbf{x}, \mathbf{p}\} = \mathbf{1}, \quad \{t, \omega\} = 1 \quad (37)$$

$$\{\mathbf{x}, \omega\} = \boldsymbol{\alpha}, \quad \{\mathbf{x}, p_0\} = -\boldsymbol{\alpha} \quad (38)$$

$$\{\mathbf{p}, \omega\} = -\mathbf{D}, \quad \{\mathbf{p}, p_0\} = \mathbf{D} \quad (39)$$

$$\{\lambda, \omega\} = -1, \quad \{\lambda, p_0\} = 1 \quad (40)$$

$$\{p_0, \omega\} = -D_0. \quad (41)$$

The bracket $\{\mathbf{x}, \mathbf{p}\} = \mathbf{1}$ justifies Dirac's choice of the usual commutation relations after the linearization of the Hamiltonian. Looking at the Hamilton equations $d\zeta^i/d\tau = \{\zeta^i, \omega\}$ we first see that $\lambda' = -t' = 1$. Thus the variable λ is redundant and can be dismissed for the quantization. The brackets involving p_0 show that despite $H = 0$, it is $-p_0$ that plays the role of an effective Hamiltonian of the initial system. For the quantum version an obvious choice which satisfies all the commutators is clearly

$$\hat{\omega} = -i\hbar \frac{d}{dt} - \hat{p}_0. \quad (42)$$

Therefore the time evolution of the quantum state given by $\hat{\omega}|\psi(t)\rangle = 0$ leads to

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \left(\boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - e\hat{\mathbf{A}}) + \beta m + e\hat{\phi} \right) |\psi(t)\rangle, \quad (43)$$

which is nothing else but the Dirac equation of a relativistic quantum particle in an electromagnetic field. This achieves our goal of deriving the relativistic Schrödinger equation, i.e., the Dirac equation from the canonical quantization of the classical relativistic Lagrangian. This quantization method could be generalized to a particle moving in a curved space and more generally to the case of time invariant reparametrization systems.

Conclusion. An extension of the Faddeev-Jackiw method in order to solve the problem of time dependent constraints has been considered. For that purpose a time parameter is introduced to treat the time as a dynamically variable, which is accompanied by the emergence of gauge symmetry. This one is fixed with the help of a supplementary variable that plays the role of a conserved Hamiltonian. After obtaining the correct brackets, we were able to give the most general form of the quantum (Schrödinger) equation, valid also for singular non-autonomous systems. The method can be naturally applied to the case of a relativistic particle in an external electromagnetic field. The theory developed in this paper should be useful for the quantization of physical constrained systems in the presence of time-dependent external fields.

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- [12] Rigorously $\hat{\varepsilon} = -i\hbar d/dt + \hat{\sigma}$ with $[\hat{t}, \hat{\sigma}] = 0$. But in this case Eq. (25) becomes $i\hbar \frac{d}{dt} |\psi\rangle = \hat{H}_{eff} |\psi(t)\rangle$ with $\hat{H}_{eff} = \hat{H} + \hat{\sigma}$. Additionally Eq. (23) becomes $d\hat{\xi}^i/dt = (f_{ij}^{-1} \frac{\partial a_j}{\partial t})_{\xi=\hat{\xi}} - [\hat{\xi}^i, \hat{\sigma}]$. Since for $\partial a_j/\partial t = 0$ we have $d\hat{\xi}^i/dt = 0$ in the Schrödinger picture, we put $\hat{\sigma} = 0$.